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Universal regularization prescription for Lovelock AdS gravity

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Abstract: A definite form for the boundary term that produces the finiteness of both the conserved quantities and Euclidean action for any Lovelock gravity with AdS asymptotics is presented. This prescription merely tells even from odd bulk dimensions, regardless the particular theory considered, what is valid even for Einstein-Hilbert and Einstein-Gauss-Bonnet AdS gravity. The boundary term is a given polynomial of the boundary extrinsic and intrinsic curvatures (also referred to as Kounterterms series). Only the coupling constant of the boundary term changes accordingly, such that it always preserves a well-posed variational principle for boundary conditions suitable for asymptotically AdS spaces. The background-independent conserved charges associated to asymptotic symmetries are found. In odd bulk dimensions, this regularization produces a generalized formula for the vacuum energy in Lovelock AdS gravity. The standard entropy for asymptotically AdS black holes is recovered directly from the regularization of the Euclidean action, and not only from the first law of thermodynamics associated to the conserved quantities.

Keywords: AdS-CFT Correspondence, Black Holes in String Theory, Classical Theories of Gravity.

Contents

1. Introduction

It is believed that the Einstein-Hilbert action is just the first term in the derivative expansion in a low energy effective theory. In general, higher order quantum corrections to gravity might appear, whose corresponding couplings are unknown until now. Among the higher derivative gravity theories, Lovelock gravity [1] possesses some special features: it leads to field equations which are up to and linear in second derivatives of the metric, it obeys generalized Bianchi identities which ensure energy conservation, and it is known to be free of ghosts when expanded on a flat space, avoiding problems with unitarity [2].

In presence of cosmological constant, the Euclidean continuation of the bulk gravity action and the conserved quantities are in general divergent. In the AdS/CFT context [3], one deals with the regularization problem for Einstein-Hilbert action by adding local functionals of the boundary metric (Dirichlet counterterms) [4]. Because of this dependence, they preserve a well-defined variational action principle for a Dirichlet boundary condition on the metric (achieved through the Gibbons-Hawking term) when varied. However, the systematic construction [5] that provides the form of the counterterms becomes cumbersome for high enough dimensions, what has prevented from finding a general pattern for the series for any dimension until now. In Lovelock gravity, it is expected that the holographic renormalization procedure would be even more complicated.

An alternative regularization scheme has been proposed for Einstein-Hilbert [6, 7], Einstein-Gauss-Bonnet [8], and Chern-Simons [9] gravity theories with AdS asymptotics. It considers the addition to the bulk action of boundary terms with dependence on the extrinsic curvature. In this paper, we show that this prescription is universal for all Lovelock-AdS theories, attaining a regularized action and finiteness of the conserved charges .

2. Lovelock gravity

In $D = d+1$ dimensions, the Lovelock action reads

$$I_D = \frac{1}{16\pi G_D} \int_M \sum_{p=0}^{[(D-1)/2]} \alpha_p L_p + c_d \int_{\partial M} B_d, \tag{2.1}$$

where L_p corresponds to the dimensional continuation of the Euler term in $2p$ dimensions

$$L_p = \frac{1}{(D-2p)!} \epsilon_{A_1 \dots A_D} \hat{\mathcal{R}}^{A_1 A_2} \dots \hat{\mathcal{R}}^{A_{2p-1} A_{2p}} e^{A_{2p+1}} \dots e^{A_D}, \tag{2.2}$$

$$= \frac{1}{2^p} \sqrt{-\mathcal{G}}\, \delta^{[\nu_1 \cdots \nu_{2p}]}_{[\mu_1 \cdots \mu_{2p}]} \hat{R}^{\mu_1 \mu_2}_{\nu_1 \nu_2} \cdots \hat{R}^{\mu_{2p-1} \mu_{2p}}_{\nu_{2p-1} \nu_{2p}} \, d^D x\,. \tag{2.3}$$

Hatted curvatures stand for D-dimensional ones. The orthonormal vielbein $e^A = e^A{}_\mu dx^\mu$ produces the spacetime metric by $\mathcal{G}_{\mu\nu} = \eta_{\rm AB}\, e^A{}_\mu e^B{}_\nu$ and the curvature 2-form is defined as $\hat{\mathcal{R}}^{\rm AB} = d\omega^{\rm AB} + \omega^A{}_C \omega^{\rm CB}$ in terms of the spin connection one-form $\omega^{\rm AB} = \omega^{\rm AB}{}_\mu dx^\mu$ and related to the spacetime Riemman tensor by $\hat{\mathcal{R}}^{\rm AB} = \frac{1}{2} \hat{R}^{\kappa\lambda}_{\mu\nu} e^A{}_\kappa e^B{}_\lambda dx^\mu dx^\nu$. Wedge products are omitted throughout. The first term in the Lovelock series corresponds to the cosmological term $L_0 = \sqrt{-\mathcal{G}}\, d^D x$, $L_1 = \sqrt{-\mathcal{G}}\, \hat{R}\, d^D x$ is the Einstein-Hilbert term, $L_2 = \sqrt{-\mathcal{G}}\, (\hat{R}_{\mu\nu\kappa\lambda} \hat{R}^{\mu\nu\kappa\lambda} - 4 \hat{R}_{\mu\nu} \hat{R}^{\mu\nu} + \hat{R}^2)\, d^D x$ is the Gauss-Bonnet term, etc. The first coefficients are $\alpha_0 = -2\Lambda = \frac{(D-1)(D-2)}{\ell^2}$, $\alpha_1 = 1$, whereas all the other α_p's are arbitrary.

The action (2.1) appears supplemented by a boundary term B_d. We shall display below the universal form of B_d for any Lovelock theory with AdS asymptotia that regularizes both the conserved quantities and the Euclidean action.

The equation of motion for a generic Lovelock gravity (with zero torsion) is obtained varying with respect to the metric and takes the form

$$E^\nu_\mu = \sum_{p=0}^{[(D-1)/2]} \frac{\alpha_p}{2^p} \delta^{[\nu\nu_1 \cdots \nu_{2p}]}_{[\mu\mu_1 \cdots \mu_{2p}]} \hat{R}^{\mu_1 \mu_2}_{\nu_1 \nu_2} \cdots \hat{R}^{\mu_{2p-1} \mu_{2p}}_{\nu_{2p-1} \nu_{2p}} = 0. \tag{2.4}$$

The vacua of a given Lovelock theory are defined as the maximally symmetric spacetimes that are globally of constant curvature. We will assume that all the corresponding cosmological constants are real and negative, i.e.,

$$\Lambda_{\mathit{eff}} = -\frac{(D-1)(D-2)}{2\ell_{\mathit{eff}}^2}, \tag{2.5}$$

where ℓ_{eff} is defined as the effective AdS radius given by the solutions to the equation

$$\sum_{p=0}^{[(D-1)/2]} \frac{\alpha_p}{(D-2p-1)!} \left(-\ell_{\mathit{eff}}^{-2} \right)^p = 0. \tag{2.6}$$

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In the present paper, we will consider spacetimes whose asymptotic behavior tends to the one of a locally AdS space, described in terms of its curvature by the condition

$$\hat{R}^{\kappa\lambda}_{\mu\nu} + \frac{1}{\ell^2_{\mathit{eff}}}\delta^{[\kappa\lambda]}_{[\mu\nu]} = 0 \tag{2.7}$$

at the boundary ∂M, or equivalently, $\hat{\mathcal{R}}^{\mathrm{AB}} + (e^A e^B)/\ell^2_{\mathit{eff}} = 0$ in differential forms language. It is important to stress that this is a generic (local) condition that does not fix completely the form of the metric.

In principle, it is not clear whether the holographic renormalization procedure might provide a systematic algorithm to regularize a generic Lovelock-AdS theory, because of the increasing complexity of the field equations respect to the Einstein-Hilbert case. The alternative construction in this paper represents a way of circumventing the difficulties of the standard method because, as we shall see below, it does not make use of the full expansion of the asymptotic metric. Indeed, we will only consider the leading order for the fields induced by this expansion to identify suitable boundary conditions for the variational problem in AAdS gravity.

Without loss of generality, we write down the line element in Gauss-normal coordinates

$$ds^2 = N^2(\rho)d\rho^2 + h_{ij}(x,\rho)dx^i dx^j, \tag{2.8}$$

that can be obtained from a generic radial ADM foliation by gauge-fixing the shift functions $N^i = 0$. A definite choice of the lapse and the boundary metric generically describes AAdS spaces in Lovelock gravity. Indeed, taking the lapse and the boundary metric as

$$N = \ell_{\mathit{eff}}/2\rho, \tag{2.9}$$

$$h_{ij} = g_{ij}(x,\rho)/\rho, \tag{2.10}$$

where $g_{ij}(x,\rho)$ accepts a regular Fefferman-Graham expansion [10]

$$g_{ij}(x,\rho) = g_{(0)ij}(x) + \rho\, g_{(1)ij}(x) + \rho^2\, g_{(2)ij}(x) + \cdots \tag{2.11}$$

identically satisfies the condition (2.7) at the conformal boundary $\rho = 0$. Here, $g_{(0)}$ is the boundary data of an initial-value problem, governed by the equations of motion written in the frame (2.8)–(2.10). However, even for Einstein-Hilbert theory, solving the coefficients $g_{(k)}$ in series (2.11) as covariant functionals of $g_{(0)}$ is only possible for low enough dimensions. Moreover, for theories where eq. (2.6) has a single root, the equations of motion posses a multiple zero in a unique AdS vacuum [11, 12]. This causes the first nontrivial relation for a given coefficient $g_{(k)}$ to appear at a higher order in ρ, what substantially increases the complexity of the equations. Therefore, one can expect that the extreme nonlinearity of the field equations in Lovelock-AdS gravity would turn impractical the application of holographic renormalization method to this class of theories.

In what follows, we propose a universal form of the boundary terms that make both the conserved charges and the Euclidean action finite in Lovelock-AdS gravity. This construction does not make use of the full Fefferman-Graham form of the metric (2.8)–(2.11) for AAdS spacetimes, but simply considers the leading-order terms in the expansion of the relevant fields.

3. $D = 2n + 1$ dimensions

In Einstein-Hilbert-AdS gravity, the standard regularization using Dirichlet counterterms reveals some differences between odd and even-dimensional cases. Indeed, it is only in odd (bulk) dimensions that a vacuum energy for AdS spacetime appears. The quasilocal stress tensor derived from the regularized action features a trace anomaly only in odd dimensions, as well, what can be traced back to a logarithmic contribution in the FG expansion (2.11).

In the alternative regularization known as Kounterterms method, the existence of a vacuum energy for Einstein-Hilbert [7] and Einstein-Gauss-Bonnet [8] AdS gravity in odd dimensions is a consequence of a different form of the boundary terms respect the even-dimensional case. Ultimately, the difference in the prescription for the regularizing boundary terms is linked to the existence of topological invariants of the Euler class whose construction is only possible in even dimensions [13, 14].

The standard Dirichlet counterterms consider the addition to the action of local, covariant functional of the boundary metric h_{ij} and the intrinsic curvature $R^{kl}_{ij}(h)$. In the present formulation, the boundary term B_d in eq. (2.1) will depend also on the extrinsic curvature K_{ij}, defined in the frame (2.8) by

$$K_{ij} = -\frac{1}{2N}\partial_\rho h_{ij}\,, \tag{3.1}$$

and, because of this dependence, we will refer to it as *Kounterterms* series.

The explicit form the Kounterterms B_{2n} adopt in any odd-dimensional Lovelock-AdS gravity can be written in a compact way as

$$\begin{aligned} B_{2n} = 2n\sqrt{-h}\int_0^1 dt\int_0^t ds\, \delta^{[i_1\cdots i_{2n-1}]}_{[j_1\cdots j_{2n-1}]} K^{j_1}_{i_1}\left(\frac{1}{2}R^{j_2j_3}_{i_2i_3} - t^2K^{j_2}_{i_2}K^{j_3}_{i_3} + \frac{s^2}{\ell^2_{eff}}\delta^{j_2}_{i_2}\delta^{j_3}_{i_3}\right)\times\cdots \\ \cdots\times\left(\frac{1}{2}R^{j_{2n-2}j_{2n-1}}_{i_{2n-2}i_{2n-1}} - t^2K^{j_{2n-2}}_{i_{2n-2}}K^{j_{2n-1}}_{i_{2n-1}} + \frac{s^2}{\ell^2_{eff}}\delta^{j_{2n-2}}_{i_{2n-2}}\delta^{j_{2n-1}}_{i_{2n-1}}\right) d^{2n}x, \end{aligned} \tag{3.2}$$

which, when expanded, produces a polynomial in the intrinsic and extrinsic curvatures whose relative coefficients are obtained performing the above parametric integrations

$$B_{2n} = n!\,\sqrt{-h}\sum_{p=0}^{n-1}\frac{(2n-2p-3)!!}{\ell^{2(n-1-p)}}b^{(p)}_{2n}\,, \tag{3.3}$$

where

$$b^{(p)}_{2n} = \delta^{[i_1\cdots i_{2p+1}]}_{[j_1\cdots j_{2p+1}]}\sum_{q=0}^{p}\frac{(-1)^{p-q}}{(p-q)!\,q!}\frac{2^{n-(p+q+1)}}{n-q}R^{j_1j_2}_{i_1i_2}\cdots R^{j_{2q-1}j_{2q}}_{i_{2q-1}i_{2q}}K^{j_{2q+1}}_{i_{2q+1}}\cdots K^{j_{2p+1}}_{i_{2p+1}}\,. \tag{3.4}$$

The tensorial formula of the boundary terms (3.2), adapted to a radial foliation of the spacetime, can be cast into a fully Lorentz-covariant $2n-$form with the definition of the second fundamental form $\theta^{AB} = n^AK^B - n^BK^A$,

$$B_{2n} = 2n\int_0^1 dt\int_0^t ds\,\epsilon_{a_1\ldots a_{2n}}K^{a_1}e^{a_2}\left(\mathcal{R}^{a_3a_4}-t^2K^{a_3}K^{a_4}+\frac{s^2}{\ell_{\mathit{eff}}^2}e^{a_3}e^{a_4}\right)\times\cdots$$

$$\cdots\times\left(\mathcal{R}^{a_{2n-1}a_{2n}}-t^2K^{a_{2n-1}}K^{a_{2n}}+\frac{s^2}{\ell_{\mathit{eff}}^2}e^{a_{2n-1}}e^{a_{2n}}\right), \tag{3.5}$$

$$= n\int_0^1 dt\int_0^t ds\,\epsilon_{A_1\ldots A_{2n+1}}\theta^{A_1A_2}e^{A_3}\left(\mathcal{R}^{A_4A_5}+t^2\theta^{A_4}_{\ \ C}\theta^{CA_5}+\frac{s^2}{\ell_{\mathit{eff}}^2}e^{A_4}e^{A_5}\right)\times\cdots$$

$$\cdots\times\left(\mathcal{R}^{A_{2n}A_{2n+1}}+t^2\theta^{A_{2n}}_{\ \ F}\theta^{FA_{2n+1}}+\frac{s^2}{\ell_{\mathit{eff}}^2}e^{A_{2n}}e^{A_{2n+1}}\right), \tag{3.6}$$

where the extrinsic curvature $K^A = K^A_B e^B$ satisfies $K_{\rm AB} = -h^C_A h^D_B n_{C;D}$, with n^A the outward unit normal vector at the boundary. The orthonormal frame takes the block-diagonal form $e^1 = Nd\rho$, $e^a = e^a_{\ i}dx^i$, such that the only non-vanishing components of $\theta^{\rm AB}$ are $\theta^{1a} = K^a = K^i_j e^a_{\ i}dx^j$, and the submanifold Levi-Civita tensor is $\epsilon_{a_1\ldots a_d} = \epsilon_{1a_1\ldots a_d}$. $\mathcal{R}^{\rm AB}$ is the intrinsic curvature 2-form, that for a radial foliation contains only components on the boundary submanifold. Remarkably, the form of B_{2n} is preserved regardless the particular theory considered, only the corresponding coupling constant changes accordingly, as shown below.

3.1 Variational principle and boundary conditions

An arbitrary variation of the action produces the equations of motion plus contributions to the surface term that can be traced back to the bulk and boundary terms in (2.1)

$$\delta I_{2n+1} = \int_M (E.O.M.)+\frac{1}{8\pi G_D}\int_{\partial M}\sum_{p=1}^{n}\frac{p\alpha_p}{(D-2p)!}\,\epsilon_{a_1..a_{2n}}\delta K^{a_1}\hat{\mathcal{R}}^{a_2a_3}\ldots\hat{\mathcal{R}}^{a_{2p-2}a_{2p-1}}e^{a_{2p}}\ldots e^{a_{2n}}$$

$$+2nc_{2n}\int_{\partial M}\int_0^1 dt\,\epsilon_{a_1..a_{2n}}\delta K^{a_1}e^{a_2}\left(\hat{\mathcal{R}}^{a_3a_4}+\frac{t^2}{\ell_{\mathit{eff}}^2}e^{a_3}e^{a_4}\right)\ldots\left(\hat{\mathcal{R}}^{a_{2n-1}a_{2n}}+\frac{t^2}{\ell_{\mathit{eff}}^2}e^{a_{2n-1}}e^{a_{2n}}\right)$$

$$-2nc_{2n}\int_{\partial M}\int_0^1 dt\,t\,\epsilon_{a_1..a_{2n}}(\delta K^{a_1}e^{a_2}-K^{a_1}\delta e^{a_2})\left(\mathcal{R}^{a_3a_4}-t^2K^{a_3}K^{a_4}+\frac{t^2}{\ell_{\mathit{eff}}^2}e^{a_3}e^{a_4}\right)\times\cdots$$

$$\cdots\times\left(\mathcal{R}^{a_{2n-1}a_{2n}}-t^2K^{a_{2n-1}}K^{a_{2n}}+\frac{t^2}{\ell_{\mathit{eff}}^2}e^{a_{2n-1}}e^{a_{2n}}\right). \tag{3.7}$$

Here, we have extensively used the Gauss-Coddazzi relation for the boundary components of the Riemann 2-form

$$\hat{\mathcal{R}}^{ab} = \mathcal{R}^{ab}-K^aK^b, \tag{3.8}$$

that in the standard tensorial notation reads

$$\hat{R}^{kl}_{ij} = R^{kl}_{ij} - K^k_iK^l_j + K^k_jK^l_i. \tag{3.9}$$

A well-defined action principle for Lovelock-AdS gravity amounts to the on-shell cancellation of the surface term in eq. (3.7) by imposing suitable boundary conditions, that either are derived from, or at least, are compatible with the asymptotic behavior of the metric (2.8)–(2.11).

For the extrinsic curvature, the FG expansion produces

$$K_j^i = h^{ik}K_{kj} = \frac{1}{\ell_{eff}}\delta_j^i - \frac{\rho}{\ell_{eff}}\left(g_{(0)}^{-1}g_{(1)}\right)_j^i - \frac{\rho^2}{\ell_{eff}}\left(2g_{(0)}^{-1}g_{(2)} - g_{(0)}^{-1}g_{(1)}g_{(0)}^{-1}g_{(1)}\right)_j^i + \cdots , \quad (3.10)$$

where the indices at the r.h.s. of the above equation are raised with the conformal structure $g_{(0)}^{ij}$. Then, the extrinsic curvature on the boundary is finite

$$K_j^i = \frac{1}{\ell_{eff}}\delta_j^i. \quad (3.11)$$

In any gravity theory, h_{ij} and K_{ij} are independent variables, because the extrinsic curvature defines the canonical momentum π^{ij}. The fact that the extrinsic curvature can be written in terms of the coefficients $g_{(k)}$ in the expansion of the metric does not mean that it is determined only by the metric $g_{(0)}$. Indeed, as it is well-known, not even h_{ij} is completely determined by solving the second-order field equations with only $g_{(0)}$ as the initial data (Fefferman-Graham ambiguity for the coefficient $g_{(n)}$ with $n = [D/2]$). Then, K_{ij} remains as an independent variable even though the first terms in the expansion (3.10) are fixed by $g_{(0)}$.

For the variational problem in odd dimensions, we will consider that at the boundary the variations obey

$$\delta K_j^i = 0, \quad (3.12)$$

that is a regular boundary condition compatible with fixing the conformal metric $g_{(0)}$ on ∂M [15, 7]. Therefore, this boundary condition does not spoil the AdS/CFT interpretation of the conformal structure $g_{(0)}$ as a given data for the holographic reconstruction of the spacetime in the gravity side, and whose dual CFT on the boundary does not have gravitational degrees of freedom.

The last line in (3.7) is identically canceled by the conditions (3.11), (3.12). The asymptotic behavior (2.7) for the curvature determines the coupling constant c_{2n}

$$c_{2n} = \frac{1}{16\pi n G_D}\left[\int_0^1 dt\,(t^2-1)^{n-1}\right]^{-1}\sum_{p=1}^{n}\frac{(-1)^p p\,\alpha_p}{(D-2p)!}\ell_{eff}^{2(n-p)}, \quad (3.13)$$

in order to cancel the rest of the surface term (3.7).

In the standard Dirichlet regularization for AdS gravity, fixing the conformal structure $g_{(0)ij}$ in the boundary metric (2.10), (2.11) will require the addition of counterterms to cancel the divergence at the boundary $\rho = 0$ [16]. In our case, we select the boundary conditions (2.7), (3.11) and (3.12), which are regular on the asymptotic region, such that the regularization process is encoded in the boundary terms already present and there is no need of further addition of counterterms.

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3.2 Conserved quantities and vacuum energy

The Noether theorem applied to Lovelock-AdS gravity states that there is a set of conserved charges $Q(\xi)$ associated to asymptotic Killing vectors ξ, that are defined as $(D-2)-$forms, and therefore, are integrated on the boundary of a spatial section at constant time. More precisely, we take a timelike ADM foliation for the line element at the boundary

$$h_{ij}\,dx^i dx^j = -\tilde{N}^2(t)\,dt^2 + \sigma_{\underline{m}\,\underline{n}}(d\varphi^{\underline{m}} + \tilde{N}^{\underline{m}}\,dt)(d\varphi^{\underline{n}} + \tilde{N}^{\underline{n}}\,dt)\,, \tag{3.14}$$

with the coordinates $x^i = (t, \varphi^{\underline{m}})$, that is defined by the unit normal vector $u_i = (-\tilde{N}, \vec{0})$. The charges are then given as the integration on the boundary Σ of a spatial section, parameterized by $\varphi^{\underline{m}}$

$$Q(\xi) = \int_\Sigma d^{D-2}\varphi\,\sqrt{\sigma}\,u_j Q_i^j\,\xi^i\,. \tag{3.15}$$

In the above formula, σ denotes the determinant of the metric $\sigma_{\underline{m}\,\underline{n}}$, related to h by $\sqrt{-h} = \tilde{N}\sqrt{\sigma}$, and ξ^i is an asymptotic Killing vector. In odd dimensions, the expression for the integrand appears naturally split in two pieces

$$Q_i^j = q_i^j + q_{(0)i}^j\,, \tag{3.16}$$

with

$$q_i^j = \frac{1}{2^{n-2}}\,\delta^{[jj_2\cdots j_{2n}]}_{[i_1 i_2\cdots i_{2n}]}\,K_i^{i_1}\delta_{j_2}^{i_2}\left[\frac{1}{16\pi G_D}\sum_{p=1}^{n}\frac{p\alpha_p}{(D-2p)!}\hat{R}^{i_3 i_4}_{j_3 j_4}\cdots\hat{R}^{i_{2p-1}i_{2p}}_{j_{2p-1}j_{2p}}\,\delta^{[i_{2p+1}i_{2p+2}]}_{[j_{2p+1}j_{2p+2}]}\cdots\delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]}\right.$$
$$\left. + n c_{2n}\int_0^1 dt\left(\hat{R}^{i_3 i_4}_{j_3 j_4} + \frac{t^2}{\ell_{\mathit{eff}}^2}\delta^{[i_3 i_4]}_{[j_3 j_4]}\right)\cdots\left(\hat{R}^{i_{2n-1}i_{2n}}_{j_{2n-1}j_{2n}} + \frac{t^2}{\ell_{\mathit{eff}}^2}\delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]}\right)\right], \tag{3.17}$$

$$q_{(0)i}^j = -\frac{n c_{2n}}{2^{n-2}}\int_0^1 dt\,t\,\delta^{[jj_2\cdots j_{2n}]}_{[i_1 i_2\cdots i_{2n}]}(\delta_{j_2}^{i_2}K_i^{i_1} + \delta_i^{i_2}K_{j_2}^{i_1})\left(R^{i_3 i_4}_{j_3 j_4} - t^2 K^{[i_3 i_4]}_{[j_3 j_4]} + \frac{t^2}{\ell_{\mathit{eff}}^2}\delta^{[i_3 i_4]}_{[j_3 j_4]}\right)\times\cdots$$
$$\cdots\times\left(R^{i_{2n-1}i_{2n}}_{j_{2n-1}j_{2n}} - t^2 K^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]} + \frac{t^2}{\ell_{\mathit{eff}}^2}\delta^{[i_{2n-1}i_{2n}]}_{[j_{2n-1}j_{2n}]}\right), \tag{3.18}$$

where we have used the shorthand $K^{[ik]}_{[jl]} = K_j^i K_l^k - K_l^i K_j^k$.

Equation (3.16) defines a splitting of the charges

$$Q(\xi) = q(\xi) + q_0(\xi)\,, \tag{3.19}$$

where

$$q(\xi) = \int_\Sigma d^{D-2}\varphi\sqrt{\sigma}\,u_j q_i^j\,\xi^i \tag{3.20}$$

will provide the mass and angular momentum for AAdS black hole solutions in Lovelock gravity. It can be shown that eq. (3.17) can be factorized in any odd dimension as

$$q_i^j = \frac{1}{2^{n-2}}\delta^{[jj_2\cdots j_{2n}]}_{[i_1 i_2\cdots i_{2n}]}K_i^{i_1}\delta_{j_2}^{i_2}\left(\hat{R}^{i_3 i_4}_{j_3 j_4} + \frac{1}{\ell_{\mathit{eff}}^2}\delta^{[i_3 i_4]}_{[j_3 j_4]}\right)\mathcal{P}^{i_5\cdots i_{2n}}_{j_5\cdots j_{2n}}, \tag{3.21}$$

where $\mathcal{P}$ is a Lovelock-type polynomial of $(n-2)-$degree in the Riemann tensor $\hat{R}^{ij}_{kl}$ and the antisymmetrized Kronecker delta $\delta^{[ij]}_{[kl]}$

$$\mathcal{P}^{i_5\ldots i_{2n}}_{j_5\ldots j_{2n}} = \sum_{p=0}^{n-2} \left(nc_{2n}\frac{D_p}{\ell_{\mathit{eff}}^{2p}} + \frac{F_p}{16\pi G_D} \right) \hat{R}^{i_5 i_6}_{j_5 j_6} \ldots \hat{R}^{i_{2(n-p)-1} i_{2(n-p)}}_{j_{2(n-p)-1} j_{2(n-p)}} \delta^{[i_{2(n-p)+1} i_{2(n-p+1)}]}_{[j_{2(n-p)+1} j_{2(n-p+1)}]} \ldots \delta^{[i_{2n-1} i_{2n}]}_{[j_{2n-1} j_{2n}]}, \tag{3.22}$$

with the coefficients of the expansion given by

$$D_p = \sum_{q=0}^{p} \frac{(-1)^{p-q}}{2q+1} \binom{n-1}{q}, \quad F_p = \sum_{q=0}^{p} \frac{(-1)^{p-q}\,(n-q)\,\alpha_{n-q}}{(2q+1)!\;\ell_{\mathit{eff}}^{2(p-q)}}. \tag{3.23}$$

As the tensorial combination $\hat{R}^{kl}_{ij} + \frac{1}{\ell_{\mathit{eff}}^2}\delta^{[kl]}_{[ij]}$ is a part of the curvature of the AdS group with an effective radius ℓ_{eff}, the factorization (3.21) implies that the charge $q(\xi)$ vanishes identically for global AdS spacetime. Note that for Einstein-Hilbert-AdS gravity, $F_p = 0$ and the above expression for $q(\xi)$ recovers the corresponding charge in [7]. As a consequence, the quantity $q_0(\xi)$

$$q_0(\xi) = \int_\Sigma d^{D-2}\varphi\,\sqrt{\sigma}\,u_j q^j_{(0)i}\,\xi^i\,, \tag{3.24}$$

is truly a tensorial formula for the vacuum energy for AAdS spacetimes in Lovelock gravity, inexistent in previous literature.

A static black hole solution for Lovelock gravity (2.1) for both odd and even dimensions D is given by the metric

$$ds^2 = -\Delta^2(r)dt^2 + \frac{dr^2}{\Delta^2(r)} + r^2\gamma_{\underline{m}\,\underline{n}}d\varphi^{\underline{m}}d\varphi^{\underline{n}}, \tag{3.25}$$

with $\Delta(r)$ given by

$$\sum_{p=1}^{[(D-1)/2]} \frac{\alpha_p}{(D-2p-1)!}\left(\frac{k-\Delta^2}{r^2}\right)^p = \frac{2\Lambda}{(D-1)!} + \frac{\mu}{(D-3)!\,r^{D-1}}, \tag{3.26}$$

where μ appears as an integration constant in the first integral of the rr component of the Lovelock equations of motion (2.4)

$$\frac{(\Delta^2)'}{r}\sum_{p=1}^{[(D-1)/2]} \frac{p\alpha_p}{(D-2p-1)!}\left(\frac{k-\Delta^2}{r^2}\right)^{p-1} - \sum_{p=1}^{[(D-2)/2]} \frac{\alpha_p}{(D-2p-2)!}\left(\frac{k-\Delta^2}{r^2}\right)^p = -\frac{2\Lambda}{(D-2)!}\,, \tag{3.27}$$

and the prime stands for the derivative with respect to r coordinate. The metric $\gamma_{\underline{m}\,\underline{n}}$ $(\underline{m},\underline{n} = 1,\ldots,D-2)$ defines the line element of the transversal section Σ^k_{D-2} whose curvature is a constant $k = \pm 1, 0$. Black hole solution (3.25) possesses an event horizon r_+, which is the largest root of the equation $\Delta^2(r_+) = 0$. For this configuration, the only non-vanishing components of the extrinsic curvature are

$$K^t_t = -\Delta'\,, \quad K^{\underline{n}}_{\underline{m}} = -\frac{\Delta}{r}\delta^{\underline{n}}_{\underline{m}}\,, \tag{3.28}$$

whereas the intrinsic curvature is

$$R^{\underline{m}_1\underline{n}_1}_{\underline{m}_2\underline{n}_2} = \frac{k}{r^2}\delta^{[\underline{m}_1\underline{n}_1]}_{[\underline{m}_2\underline{n}_2]}\,, \tag{3.29}$$

which, in turn, produces the boundary components of the Riemann tensor to be

$$\hat{R}^{t\underline{n}}_{t\underline{m}} = -\frac{(\Delta^2)'}{2r}\delta^{\underline{n}}_{\underline{m}}\,, \qquad \hat{R}^{\underline{m}_1\underline{n}_1}_{\underline{m}_2\underline{n}_2} = \frac{k-\Delta^2}{r^2}\delta^{[\underline{m}_1\underline{n}_1]}_{[\underline{m}_2\underline{n}_2]}\,. \tag{3.30}$$

Differentiating eq. (3.26) with respect to the horizon radius r_+ and combining eqs. (3.26), (3.27), we obtain the relation

$$\frac{\partial\mu}{\partial r_+} = (D-3)!\,(\Delta^2)'|_{r_+}\sum_{p=1}^{[(D-1)/2]}\frac{p\alpha_p}{(D-2p-1)!}\,r_+^{D-2p-1}k^{p-1}. \tag{3.31}$$

From the equation (3.26) that dictates the form of the function $\Delta^2(r)$ in the metric, we can obtain the asymptotic behavior ($r\to\infty$)

$$\Delta^2(r) \approx k + \frac{r^2}{\ell^2_{\mathit{eff}}} - \frac{\mu}{(D-3)!}\left[\sum_{p=1}^{[(D-1)/2]}\frac{p\alpha_p(-\ell^{-2}_{\mathit{eff}})^{p-1}}{(D-2p-1)!}\right]^{-1}\frac{1}{r^{D-3}} + \cdots \tag{3.32}$$

The corresponding *mass* is given by the evaluation of eq. (3.20) for the timelike Killing vector $\xi = \partial_t$

$$q(\partial_t) = M = (D-2)!\,vol(\Sigma^k_{D-2})(\Delta^2)'\left[\frac{1}{16\pi G_D}\sum_{p=1}^{n}\frac{p\alpha_p}{(D-2p)!}r^{D-2p}(k-\Delta^2)^{p-1}\right.$$
$$\left.\left. + nc_{2n}r\int_0^1 dt\left(k-\Delta^2+\frac{t^2r^2}{\ell^2_{\mathit{eff}}}\right)^{n-1}\right]\right|^{\infty}. \tag{3.33}$$

Using the asymptotic form of $\Delta^2(r)$ from (3.32), we see that the divergent terms $O(r^{D-1})$ in the evaluation of the above formula exactly cancel out and one gets the finite result

$$M = \frac{(D-2)\,vol(\Sigma^k_{D-2})}{16\pi G_D}\mu. \tag{3.34}$$

The zero-point *(vacuum)* energy is then given by (3.24) as

$$q_0(\partial_t) = E_0 = 2nc_{2n}(D-2)!\,vol(\Sigma^k_{D-2})\left(\Delta^2 - \frac{r(\Delta^2)'}{2}\right)\int_0^1 dt\,t\left(k - t^2\Delta^2 + \frac{t^2r^2}{\ell^2_{\mathit{eff}}}\right)^{n-1}\Bigg|^{\infty}, \tag{3.35}$$

that can be worked out using the asymptotic form (3.32), giving the finite result

$$E_0 = (-k)^n\frac{vol(\Sigma^k_{D-2})}{16\pi nG_D}(2n-1)!!^2\sum_{p=1}^{n}\frac{(-1)^{p-1}p\alpha_p}{(D-2p)!}\ell^{2n-2p}_{\mathit{eff}}. \tag{3.36}$$

3.3 Black hole entropy

The Euclidean period β is defined as the inverse of black hole temperature T such that in the Euclidean sector the solution (3.25) does not have a conical singularity at the horizon. In doing so, one obtains $\beta = 4\pi/(\Delta^2)'|_{r_+}$. The black hole entropy S is defined in the canonical ensemble (the surface gravity is kept fixed at the horizon) as

$$S = I^E + \beta\mathcal{E}, \tag{3.37}$$

in terms of the total Euclidean action I^E and the thermodynamical energy

$$\mathcal{E} = -\frac{\partial I^E}{\partial \beta} \tag{3.38}$$

of the black hole. The Euclidean bulk action is evaluated for a static black hole of the form (3.25) as

$$I^E_{\text{bulk}} = -\frac{(D-2)!}{16\pi G_D} vol(\Sigma^k_{D-2})\,\beta \sum_{p=1}^{n} \frac{p\alpha_p}{(D-2p)!} [r^{D-2p}(\Delta^2)'\,(k-\Delta^2)^{p-1}]|^{\infty}_{r_+}, \tag{3.39}$$

and it is rendered finite by the addition of the suitable boundary term (3.2), whose evaluation in the Euclidean solution is

$$\int_{\partial M} B^E_{2n} = -n(D-2)!\,vol(\Sigma^k_{D-2})\beta \left[r(\Delta^2)' \int_0^1 dt \left(k - \Delta^2 + \frac{t^2 r^2}{\ell^2_{\mathit{eff}}}\right)^{n-1} \right.$$
$$\left. +2\left(\Delta^2 - \frac{r(\Delta^2)'}{2}\right) \int_0^1 dt\, t \left(k - t^2\Delta^2 + \frac{t^2 r^2}{\ell^2_{\mathit{eff}}}\right)^{n-1} \right] \Bigg|^{\infty} \tag{3.40}$$

Therefore, the total action contains two pieces. At $r = \infty$, the contribution from the bulk action I^E_{bulk} combines with the boundary term $c_{2n}\int_{\partial M} B^E_{2n}$ to produce $-\beta$ times the Noether charge $Q(\partial_t) = M + E_0$

$$I^E_{2n+1} = \frac{(D-2)!}{16\pi G_D} vol(\Sigma^k_{D-2}) \left[4\pi \sum_{p=1}^{n} \frac{p\alpha_p}{(D-2p)!} r_+^{D-2p} k^{p-1} - \frac{\beta\mu}{(D-3)!} \right.$$
$$\left. -\beta(-k)^n \frac{(2n-1)!!^2}{n(D-2)!} \sum_{p=1}^{n} \frac{(-1)^{p-1} p\alpha_p}{(D-2p)!} \ell_{\mathit{eff}}^{2n-2p} \right]. \tag{3.41}$$

This identification guarantees that all the divergencies at radial infinity are exactly canceled.

The definition of thermodynamic energy, using equation (3.31), gives

$$\mathcal{E} = -\frac{\partial I^E_{2n+1}/\partial r_+}{\partial\beta/\partial r_+} = M + E_0, \tag{3.42}$$

which recovers the same total energy as from the Noether charge $Q(\partial_t)$ of (3.19). As a consequence, the entropy (3.37) is simply given by the Noether charge evaluated at the horizon

$$S = \frac{(D-2)!}{4G_D} vol(\Sigma^k_{D-2}) \sum_{p=1}^{n} \frac{p\alpha_p}{(D-2p)!} r_+^{D-2p} k^{p-1}. \tag{3.43}$$

4. $D=2n$ dimensions

For even dimensions, an alternative regularization procedure was developed originally for Einstein-Hilbert-AdS action in [6] and applied to the same problem in AAdS gravity in Einstein-Gauss-Bonnet theory [8]. As we shall explicitly show below, the universal form of the boundary term that renders the conserved charges and Euclidean action finite in Lovelock-AdS gravity in $D=2n$ corresponds to the (maximal) $n-$th Chern form [17−19]

$$B_{2n-1}=n\int_0^1 dt\,\epsilon_{A_1\ldots A_{2n}}\theta^{A_1A_2}\left(\mathcal{R}^{A_3A_4}+t^2\theta^{A_3}_{\ C}\theta^{CA_4}\right)\ldots\left(\mathcal{R}^{A_{2n-1}A_{2n}}+t^2\theta^{A_{2n-1}}_{\ F}\theta^{FA_{2n}}\right). \quad (4.1)$$

Eq. (4.1) can be projected to the boundary indices to work out its equivalence in tensorial notation

$$\begin{aligned}B_{2n-1} &= 2n\int_0^1 dt\,\epsilon_{a_1..a_{2n-1}}K^{a_1}\left(\mathcal{R}^{a_2a_3}-t^2K^{a_2}K^{a_3}\right)\ldots\left(\mathcal{R}^{a_{2n-2}a_{2n-1}}-t^2K^{a_{2n-2}}K^{a_{2n-1}}\right) \quad (4.2)\\ &= 2n\sqrt{-h}\int_0^1 dt\delta^{[j_1\cdots j_{2n-1}]}_{[i_1\ldots i_{2n-1}]}K^{i_1}_{j_1}\left(\frac{1}{2}R^{i_2i_3}_{j_2j_3}-t^2K^{i_2}_{j_2}K^{i_3}_{j_3}\right)\ldots \quad (4.3)\\ &\qquad\ldots\left(\frac{1}{2}R^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}}-t^2K^{i_{2n-2}}_{j_{2n-2}}K^{i_{2n-1}}_{j_{2n-1}}\right)d^{2n-1}x.\end{aligned}$$

In the last formula, the parametric integration reflects the action of the Cartan homotopy operator, used to obtain the correction to the Euler characteristic due to the introduction of a boundary in the Euler theorem. The integral in t is a convenient shorthand, but it also generates the suitable coefficients in the binomial expansion

$$B_{2n-1}=2n\sqrt{-h}\sum_{p=0}^{n-1}\frac{(-1)^{n-p-1}}{2^p\,(2n-2p-1)}b^{(p)}_{2n-1}, \quad (4.4)$$

where

$$b^{(p)}_{2n-1}=\delta^{[i_1\cdots i_{2p}\cdots i_{2n-1}]}_{[j_1\cdots j_{2p}\cdots j_{2n-1}]}R^{j_1j_2}_{i_1i_2}\cdots R^{j_{2p-1}j_{2p}}_{i_{2p-1}i_{2p}}\,K^{j_{2p+1}}_{i_{2p+1}}\cdots K^{j_{2n-1}}_{i_{2n-1}}\,. \quad (4.5)$$

The surface term coming from an arbitrary on-shell variation of the action (2.1) adopts a slightly simpler form than in the odd-dimensional case

$$\begin{aligned}\delta I_{2n} &= \int_{\partial M}\frac{1}{8\pi G_D}\sum_{p=1}^{n-1}\frac{p\alpha_p}{(D-2p)!}\epsilon_{a_1\ldots a_{2n-1}}\delta K^{a_1}\hat{\mathcal{R}}^{a_2a_3}\ldots\hat{\mathcal{R}}^{a_{2p-2}a_{2p-1}}e^{a_{2p}}\ldots e^{a_{2n-1}}+\\ &\qquad +2nc_{2n-1}\epsilon_{a_1\ldots a_{2n-1}}\delta K^{a_1}\hat{\mathcal{R}}^{a_2a_3}\ldots\hat{\mathcal{R}}^{a_{2n-2}a_{2n-1}}. \quad (4.6)\end{aligned}$$

An appropriate choice of the coupling constant c_{2n-1} as

$$c_{2n-1}=-\frac{1}{16\pi nG_D}\sum_{p=1}^{n-1}\frac{p\alpha_p}{(D-2p)!}\left(-\ell^2_{\mathit{eff}}\right)^{n-p}. \quad (4.7)$$

makes the above expression vanish identically for AAdS spacetimes (2.7). The regularity of the asymptotic condition (2.7) implies that the well-defined action principle achieved in this way is also a finite one, because no additional divergences are induced by the addition of the Kounterterms (4.3).

4.1 Conserved charges

In Einstein-Hilbert and Einstein-Gauss-Bonnet with negative cosmological constant, we have seen that the addition of boundary terms with explicit dependence on the extrinsic curvature K_{ij} solve at once two problems that in general are not necessarily related: the variational principle and the finiteness of the Noether charges and Euclidean action. Whenever the action is stationary for boundary conditions compatible with the asymptotic structure of AAdS spacetimes, the theory does not require a further regularization on top of the addition of B_d in eq. (2.1).

The conserved charges constructed using the Noether theorem have the form

$$Q(\xi) = \int_{\Sigma} d^{D-2}\varphi \sqrt{\sigma}\, u_j Q_i^j\, \xi^i\,, \tag{4.8}$$

with the integrand given by

$$Q_i^j = \frac{1}{2^{n-2}}\delta^{[jj_2\cdots j_{2n-1}]}_{[i_1i_2\ldots i_{2n-1}]}K_i^{i_1}\left[\frac{1}{16\pi G_D}\sum_{p=1}^{n-1}\frac{p\alpha_p}{(D-2p)!}\hat{R}^{i_2i_3}_{j_2j_3}\ldots\hat{R}^{i_{2p-2}i_{2p-1}}_{j_{2p-2}j_{2p-1}}\delta^{[i_{2p}i_{2p+1}]}_{[j_{2p}j_{2p+1}]}\ldots\delta^{[i_{2n-2}i_{2n-1}]}_{[j_{2n-2}j_{2n-1}]}\right.$$
$$\left.+nc_{2n-1}\hat{R}^{i_2i_3}_{j_2j_3}\ldots\hat{R}^{i_{2n-2}i_{2n-1}}_{j_{2n-2}j_{2n-1}}\right]. \tag{4.9}$$

The *mass* for Lovelock-AdS black holes (3.25), (3.26) comes from the above formula for the Killing vector $\xi = \partial_t$, that is

$$Q(\partial_t) = M = (D-2)!\,vol\left(\Sigma^k_{D-2}\right)(\Delta^2)'\left[\frac{1}{16\pi G_D}\sum_{p=1}^{n-1}\frac{p\alpha_p}{(D-2p)!}r^{D-2p}(k-\Delta^2)^{p-1}\right.$$
$$\left.\left.+nc_{2n-1}(k-\Delta^2)^{n-1}\right]\right|^{\infty}. \tag{4.10}$$

As in the odd-dimensional case, taking the asymptotic expansion of the functions involved shows that the divergences at order r^{D-1} coming both from the bulk and boundary parts of the action are exactly canceled. Thus, we obtain

$$M = \frac{(D-2)vol(\Sigma^k_{D-2})}{16\pi G_D}\mu. \tag{4.11}$$

4.2 Black hole entropy

The Euclidean bulk action I^E_{bulk} is still given by the even-dimensional equivalence of equation (3.39)

$$I^E_{\text{bulk}} = -\frac{(D-2)!}{16\pi G_D}vol(\Sigma^k_{D-2})\,\beta\sum_{p=1}^{n-1}\frac{p\alpha_p}{(D-2p)!}\left[r^{D-2p}(\Delta^2)'(k-\Delta^2)^{p-1}\right]\Big|^{\infty}_{r_+},$$

while the Euclidean continuation of the boundary term takes the form

$$\int_{\partial M} B^E_{2n-1} = -n(D-2)!\,vol(\Sigma^k_{D-2})\,\beta\,(\Delta^2)'(k-\Delta^2)^{n-1}\Big|^{\infty}. \tag{4.12}$$

In the total Euclidean action in even dimensions evaluated for a black hole (3.25), (3.26)

$$I_{2n}^E = I_{\text{bulk}}^E + c_{2n-1}\int_{\partial M} B_{2n-1}^E, \tag{4.13}$$

the term at infinity corresponds to $-\beta M$, where M is the Noether mass in eqs. (4.10), (4.11), that is

$$I_{2n}^E = \frac{(D-2)!}{16\pi G_D} vol(\Sigma_{D-2}^k)\left[4\pi\sum_{p=1}^{n-1}\frac{p\alpha_p}{(D-2p)!}r_+^{D-2p}k^{p-1} - \frac{\beta\mu}{(D-3)!}\right]. \tag{4.14}$$

The consistency between the regularization procedure and the thermodynamic ensemble is corroborated by the fact that the thermodynamic energy

$$\mathcal{E} = -\frac{\partial I_{2n}^E}{\partial\beta} = M, \tag{4.15}$$

reobtains the corresponding Noether charge. Finally, the black hole entropy is expressed in terms of the horizon r_+ in a similar form as eq. (3.43) for the odd-dimensional case

$$S = \frac{(D-2)!}{4G_D}\,vol(\Sigma_{D-2}^k)\sum_{p=1}^{n-1}\frac{p\alpha_p}{(D-2p)!}r_+^{D-2p}k^{p-1}. \tag{4.16}$$

5. Particular cases

Einstein-Gauss-Bonnet-AdS gravity. In this case, all the coefficients in the Lovelock series are vanishing but $\alpha_0 = -2\Lambda$, $\alpha_1 = 1$ and $\alpha_2 = \alpha$, where α is an arbitrary positive coupling constant. The effective AdS radius is modified by the Gauss-Bonnet coupling as

$$\frac{1}{\ell_{\mathit{eff}}^2} = \frac{1 \pm \sqrt{1-4(D-3)(D-4)\alpha/\ell^2}}{2(D-3)(D-4)\alpha}, \tag{5.1}$$

such that the solutions tend asymptotically to a constant curvature spacetime with that radius. The Noether charge in the corresponding odd and even dimensions, evaluated for a timelike Killing vector $\xi = \partial_t$ for Boulware-Deser black holes

$$\Delta^2(r) = k + \frac{r^2}{2(D-3)(D-4)\alpha}\left[1 \pm \sqrt{1 - \frac{4(D-3)(D-4)\alpha}{\ell^2} + \frac{4(D-3)(D-4)\alpha\,\mu}{r^{D-1}}}\right], \tag{5.2}$$

recovers the mass obtained by background-dependent methods [20−23]

$$M = \frac{(D-2)\,vol(\Sigma_{D-2}^k)}{16\pi G_D}\mu. \tag{5.3}$$

However, background-independent methods are the only ones that can detect the presence of a vacuum energy for Einstein-Gauss-Bonnet theory. The Dirichlet regularization for arbitrary couplings of quadratic curvature terms, and therefore, useful to treat the EGB action, is only known in five dimensions [24] and, for the Gauss-Bonnet case, it has shown to be ambiguous [25]. The procedure carried out here reproduces, by direct replacement of the corresponding Lovelock coefficients $\{\alpha_0, \alpha_1, \alpha_2\}$ in eq. (3.36), the general formula for the vacuum energy for EGB-AdS theory

$$E_0 = (-k)^n \frac{vol(\Sigma^k_{D-2})}{8\pi G_D} \ell_{\mathit{eff}}^{2n-2} \frac{(2n-1)!!^2}{(2n)!} \left(1 - \frac{2\alpha}{\ell_{\mathit{eff}}^2}(D-2)(D-3)\right), \tag{5.4}$$

that was first computed in [8] using Kounterterms regularization. The form of the boundary terms that makes possible this result for EGB-AdS gravity shall be shown to be universal below because it also provides finite expressions for the conserved quantities of AAdS solutions in Lovelock gravity.

The existence of a vacuum energy does not modify the black hole entropy because as the total energy $\mathcal{E} = M + E_0$ is shifted by a constant with respect to the mass calculated with background-dependent methods, the Euclidean action changes in a consistent manner. As a consequence, the entropy of the system can be written as

$$S = \frac{vol(\Sigma^k_{D-2})\, r_+^{D-2}}{4G_D} \left[1 + \frac{2k\alpha(D-2)(D-3)}{r_+^2}\right], \tag{5.5}$$

in both odd and even dimensions. This formula have been found by several authors [26 − 28], where some of the conserved quantities, including the entropy function have been computed assuming that they satisfy the First Law of black hole thermodynamics. The same result can be derived from the regularized Euclidean action as the free energy, obtained as difference between the Euclidean bulk action evaluated for a EGB-AdS black hole and AdS vacuum [29 − 31] (for a similar background-substraction computation in string generated gravity with quadratic curvature couplings, see [32]).

Dimensionally continued gravity. If one considers that the equation of motion for Lovelock gravity (2.4) posseses m different vacuum (constant curvature) solutions, this means that $\alpha_p = 0$ for $p > m$, while $\alpha_m \neq 0$ for $1 \leq m \leq \left[\frac{D-1}{2}\right]$. Then, eq. (2.4) can also be written in the form

$$E_\mu^\nu = \delta^{[\nu\nu_1\cdots\nu_{2m}]}_{[\mu\mu_1\cdots\mu_{2m}]} \left(\hat{R}^{\mu_1\mu_2}_{\nu_1\nu_2} + \gamma_1 \delta^{[\mu_1\mu_2]}_{[\nu_1\nu_2]}\right) \cdots \left(\hat{R}^{\mu_{2m-1}\mu_{2m}}_{\nu_{2m-1}\nu_{2m}} + \gamma_m \delta^{[\mu_{2m-1}\mu_{2m}]}_{[\nu_{2m-1}\nu_{2m}]}\right) = 0, \tag{5.6}$$

where

$$\alpha_{m-p} = \alpha_m \frac{(D-2m+2p-1)!}{(D-2m-1)!} \sum_{i_1<\ldots<i_p=1}^{m} \gamma_{i_1} \ldots \gamma_{i_p}\,, \qquad 1 \leq p \leq m. \tag{5.7}$$

The relation (5.7) defines an algebraic system of m equations for m unknows $\gamma_1, \ldots, \gamma_m$. In the particular case where $\gamma_1 = \cdots = \gamma_m = \frac{1}{\ell_{\mathit{eff}}^2}$, the above equation produces for the couplings α_p the special values

$$\alpha_p = \frac{(D-2p-1)!}{(D-3)!\, m} \ell_{\mathit{eff}}^{2p-2} \binom{m}{p}, \qquad 0 \leq p \leq m. \tag{5.8}$$

In the conventions we have adopted in this paper ($\alpha_0 = -2\Lambda$), we find that the effective AdS radius is $\ell_{\mathit{eff}}^2 = \ell^2/m$, whereas equation (2.6) becomes an identity.

The label m takes the maximal value ($m = [\frac{D-1}{2}]$) for two particular Lovelock theories that feature a symmetry enhancement from Lorentz to AdS group: Chern-Simons-AdS (CS-AdS) and Born-Infeld-AdS (BI-AdS) in odd and even dimensions, respectively. Both theories posses a single cosmological constant and the maximal number of curvatures for a given dimension. Static black hole solutions for CS-AdS and BI-AdS theories were studied in [11].

CS-AdS gravity is obtained from a Chern-Simons density for the AdS group in $D=2n+1$, such that the corresponding coefficients (5.8) in eq. (2.1) are given by

$$\alpha_p^{(CS)} = \frac{(D-2p-1)!}{(D-3)!\,n}\ell_{eff}^{2p-2}\binom{n}{p}\,, \qquad 0 \leq p \leq n, \tag{5.9}$$

which produce equations of motion where AdS vacuum is a zero of $n-$th order. Topological static black holes were studied in [33]. The horizon radius is defined by the relation (3.26) $\mu = \frac{1}{n\ell_{eff}^2}(r_+^2 + k\ell_{eff}^2)^n$, such that the formula (3.34) gives

$$M^{(CS)} = \frac{(D-2)\,vol(\Sigma_{D-2}^k)}{16\pi n G_D \ell_{eff}^2}\left(r_+^2 + k\ell_{eff}^2\right)^n, \tag{5.10}$$

whereas the vacuum energy (3.36) reduces to the form

$$E_0^{(CS)} = -k^n \frac{(D-2)\,vol(\Sigma_{D-2}^k)}{16\pi n G_D}\ell_{eff}^{2(n-1)}. \tag{5.11}$$

The last expression corresponds to the energy of global AdS spacetime. In CS-AdS gravity, the AdS vacuum is separated from black holes ($M > 0$) by a mass gap of naked singularities with mass in the interval $M = (E_0, 0)$, as in $(2+1)$ dimensions. Eq. (5.10) is the standard result for the mass, found in Hamiltonian form in [11, 33]. The vacuum energy was obtained as a Noether charge evaluated in AdS in the background-independent formulation presented in [9], using a boundary term proportional to (3.2). It is remarkable that the symmetry enhancement in this case, turns the Dirichlet counterterms series exactly solvable from the divergent parts in the expansion of the canonical variation of the action [34], and this allows an explicit comparison between the Kounterterms procedure and the Dirichlet regularization [35].

As usual in Lovelock gravity, black hole entropy in CS-AdS theory cannot be related to the horizon area, but just expressed from eq. (3.43) in terms of r_+ as

$$S^{(CS)} = \frac{(D-2)\,vol(\Sigma_{D-2}^k)}{4G_D}\int_0^{r_+} dr\,(r^2 + k\ell_{eff}^2)^{n-1}. \tag{5.12}$$

The last result matches the one obtained using a mini-superspace model in the canonical formalism [11, 33], and also the prescription for the entropy as a given $(D-2)-$form integrated at the horizon [27].

For BI-AdS gravity in $D=2n$ dimensions, the couplings (5.8) become

$$\alpha_p^{(BI)} = \frac{(D-2p)!}{(D-2)!\,n}\ell_{eff}^{2p-2}\binom{n}{p}\,, \qquad 0 \leq p \leq n-1. \tag{5.13}$$

BI-AdS gravity can naturally incorporate into the bulk piece of the action (2.1) the (topological) Euler term $\mathcal{E}_{2n} = \epsilon_{A_1 \ldots A_D} \hat{\mathcal{R}}^{A_1A_2} \ldots \hat{\mathcal{R}}^{A_{2n-1}A_{2n}}$ with an appropriate weight factor $\alpha_n^{(BI)}$ arising from (5.13) for $p = n$. As the Euler term is locally equivalent to the boundary term (4.3), the complete action (2.1) is also written as

$$I_D^{(BI)} = \frac{1}{16\pi n G_D} \frac{\ell_{\mathit{eff}}^{2n-2}}{(D-2)!\,2^n} \int_M d^{2n}x \sqrt{-\mathcal{G}}\, \delta^{[\mu_1 \cdots \mu_{2n}]}_{[\nu_1 \cdots \nu_{2n}]} \left(\hat{R}^{\nu_1\nu_2}_{\mu_1\mu_2} + \frac{1}{\ell_{\mathit{eff}}^2} \delta^{[\nu_1\nu_2]}_{[\mu_1\mu_2]} \right) \cdots$$
$$\cdots \left(\hat{R}^{\nu_{2n-1}\nu_{2n}}_{\mu_{2n-1}\mu_{2n}} + \frac{1}{\ell_{\mathit{eff}}^2} \delta^{[\nu_{2n-1}\nu_{2n}]}_{[\mu_{2n-1}\mu_{2n}]} \right), \tag{5.14}$$

which is both invariant under AdS group and regularized by construction. Again, the Kounterterms procedure provides a finite answer for the mass from eq. (4.11)

$$M^{(BI)} = \frac{vol(\Sigma^k_{D-2})}{8\pi G_D \ell_{\mathit{eff}}^2} r_+ \left(r_+^2 + k\ell_{\mathit{eff}}^2 \right)^{n-1}, \tag{5.15}$$

that has been obtained in Hamiltonian way, but also in a background-independent method using the regularizing effect given by the inclusion of the Euler term [14].

The static black hole entropy in BI-AdS gravity is found by plugging the coefficients (5.13) into the formula (4.16)

$$S^{(BI)} = \frac{vol(\Sigma^k_{D-2})}{4G_D} \left[\left(r_+^2 + k\ell_{\mathit{eff}}^2 \right)^{n-1} - \left(k\ell_{\mathit{eff}}^2 \right)^{n-1} \right]. \tag{5.16}$$

The issue of the entropy for BI-AdS black holes is more subtle than the regularization of the Noether charges. If, instead, one uses the Euler term $\mathcal{E}_{2n}$ to render the Euclidean action finite as in (5.14), the entropy found will be shifted by the opposite of the last term proportional to k^{n-1} in (5.16), which is related to the Euler characteristic χ_{2n} of the manifold. For solutions with hyperbolic horizon ($k = -1$), that entropy could become negative for physically reasonable black holes ($r_+ < \ell_{\mathit{eff}}$), as noticed for Einstein-Hilbert in [6]. However, in our approach, the problem is circumvented by using the Chern form (4.3), what provides the consistent regularization prescription and the correct entropy in all cases of even-dimensional Lovelock gravity.

Lovelock unique vacuum. Extending the idea of a single cosmological constant of Dimensionally Continued AdS Gravity, one can adjust the coefficients of Lovelock series to attain a family of inequivalent gravity theories that posses a unique AdS vacuum [12]. The choice (5.8) produces equations of motion where global AdS (maximally symmetric) spacetime is a zero of $m-$th order.

The mass of asymptotically AdS static black holes was computed using Hamiltonian formalism and AdS as the natural background reference for the energy. Here, we use the background-independent formulas (3.34) and (4.11), to obtain the mass

$$M^{(LUV)} = \frac{(D-2)\, vol(\Sigma^k_{D-2})}{16\pi m G_D \ell_{\mathit{eff}}^2} r_+^{D-2m-1} \left(r_+^2 + k\ell_{\mathit{eff}}^2 \right)^m. \tag{5.17}$$

In odd dimensions, the vacuum energy can be calculated directly from eq. (3.36), and it turns to be

$$E_0^{(LUV)} = (-k)^n \frac{vol(\Sigma^k_{D-2})}{16\pi n G_D} \frac{(2n-1)!!^2}{(D-3)!} \ell_{\mathit{eff}}^{2(n-1)} \int_0^1 du\, u^{D-2m-1} (u^2-1)^{m-1}. \tag{5.18}$$

The corresponding entropy can be computed from (3.43), (4.16), once the Euclidean action has been regularized by the addition of the Kounterterms series, and takes the explicit form

$$S^{(LUV)} = \frac{(D-2)vol(\Sigma^k_{D-2})}{4G_D} \int_0^{r_+} dr\, r^{D-2m-1} \, (r^2 + k\ell_{\mathit{eff}}^2)^{m-1}. \tag{5.19}$$

In [27], the above expression was obtained from the direct application of Wald's prescription [36] for Lovelock Unique Vacuum gravity. The same results can be reproduced using identities derived from the gravitational bulk Lagrangian in [28]. Despite the fact that these approaches lead to the correct formula for the entropy in all cases, they deal only with local properties of the action at the horizon and it do not really provide an answer to the problem of bulk action regularization for the asymptotic region in AdS gravity.

It is worthwhile noticing that this set of theories is not free from the inconsistencies produced by negative values of the entropy (5.19) when the spatial section has negative curvature. In that sense, Lovelock Unique Vacuum does not feature a more sensible thermodynamic behavior than, e.g., Einstein-Gauss-Bonnet with AdS asymptotics.

The existence of different values for the vacuum energy (5.18) for a given odd dimension suggests that the set of gravity theories ranging between EH and CS should have a set of inequivalent CFT duals. This is also clear from the information coming from the Weyl anomaly. On the contrary to EH-AdS, in $(2n+1)$-dimensional CS-AdS gravity the holographic Weyl anomaly is proportional only to the Euler term in 2n-dimensions (type A anomaly) with no contributions from the Weyl tensor (type B anomaly) [34]. Then, odd-dimensional gravity theories with $1 < m < n$ should posses a combination of both types of holographic anomaly. As this information is usually extracted from the finite part of a quasilocal stress tensor for AdS gravity, the present regularization prescription for all Lovelock theories can be regarded as a step ahead towards a general formula for the holographic anomaly in Lovelock-AdS gravity.

6. Conclusions

In this paper we have provided the explicit form of the boundary terms that regularize the conserved quantities for asymptotically AdS solutions of Lovelock gravity. The prescription for the boundary terms contains the extrinsic curvature and it only distinguishes even from odd dimensions, independently of the particular model under consideration. Just the weight factor of these terms needs to be consistently tuned in order to have a well-posed variational principle for AAdS spacetimes. At the same time, the finiteness of the Euclidean action is achieved.

In all the known cases (Einstein-Gauss-Bonnet, Chern-Simons, Born-Infeld, Lovelock Unique Vacuum) both conserved charges and black hole thermodynamics agree with the standard results. Even if the Noether charges assign a non-vanishing vacuum energy to AdS in odd dimensions (which is unobservable in background-dependent methods), the entropy expression is still the correct one, because the Euclidean action appears shifted consistently.

In even dimensions the boundary prescription is given by the maximal Chern form. This is the structure appearing in the Euler theorem as the correction to the Euler characteristic of the manifold due to the boundary. In odd dimensions, the regularizing terms are linked to the existence of extensions of Chern-Simons densities called transgression forms [37].

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